Geometric Flows

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Chapter 1: Regular Surfaces

Local Parametrizations of Class C^k :

Def:: Consider a subset $M \subset \mathbb{R}^3$. A function $F(u,v) : \mathcal{U} \to \mathcal{O}$ from an open subset $\mathcal{U} \subset \mathbb{R}^2$ onto an open subset $\mathcal{O} \subset M$ is called a C^k local parametrization (or a C^k local coordinate chart) of M (where $k \geq 1$) if all of the following holds:
(1) $F : \mathcal{U} \to \mathbb{R}^3$ is C^k when the codomain is regarded as $\mathbb{R}^3$.
(2) $F : \mathcal{U} \to \mathcal{O}$ is a homeomorphism, meaning that $F : \mathcal{U} \to \mathcal{O}$ is bijective, and both $F : \mathcal{U} \to \mathcal{O}$ and $F^{-1} : \mathcal{O} \to \mathcal{U}$ are continuous.
(3) For all $(u,v) \in \mathcal{U}$, the cross product:

$$\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v} \neq 0.$$

The coordinates (u,v) are called the local coordinates of M. If $F : \mathcal{U} \to M$ is of class C^k for any integer k, then F is said to be a C^∞ (or smooth) local parametrization.

Generalized Stereographic parametrization:

We define n-dimesnion sphere as

$$S^n := (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \ldots + x_{n+1}^2 = 1$$

We can parametrize this by considering the projection onto R^n.
Let $s = \sum_{i=1}^n x_i^2$, then we define the functions

$$F_+(x_1, \ldots, x_n) := (\frac{2x_1}{s+1} + \ldots + \frac{2x_n}{s+1} + \frac{s-1}{s+1})$$

This parametrization $R^n \to R^{n+1}$ convers the whole sphere except $(0, \ldots, 0, 1)$.

$$F_-(x_1, \ldots, x_n) := (\frac{2x_1}{s+1} + \ldots + \frac{2x_n}{s+1} + \frac{-s+1}{s+1})$$

This parametrization $R^n \to R^{n+1}$ convers the whole sphere except $(0, \ldots, 0, -1)$. The inverse can be found to be

$$F_+^{-1}(x_1, \ldots, x_{n+1}) = (\frac{x_1}{1 - x_{n+1}}, \ldots, \frac{x_n}{1 - x_{n+1}})$$

$$F_-^{-1}(x_1, \ldots, x_{n+1}) = (\frac{x_1}{1 + x_{n+1}}, \ldots, \frac{x_n}{1 + x_{n+1}})$$

Transition map for this parametrization

With regards to covering the whole sphere, notice that the transition map is given by

$$F_-^{-1} \circ F_+(x_1, \ldots, x_n) = (\frac{x_1}{s}, \ldots, \frac{x_n}{s}) = F_+^{-1} \circ F_-(x_1, \ldots, x_n)$$

These map are smooth since in our domain $s \neq 0$.

Chapter 2: Abstract Manifolds

Homeomorphism

As defined in the notes, $F : U \to \mathcal{O}$ is homeomorphism if F is bijection and both F and F^{-1} are continuous.
Properties:
1. Converserves compactness. Example $(0, 1)$ is homeomorphic to $\mathbb{R}$ but $[0, 1]$ is not.
2. Open and close is conserved. Example $(0, 1)$ is not homeomorphism to $[0, 1]$.
3. Connectness is conserved. Example punctured $(0, 2\pi)$ is not homeomorphic to a punctured circle (since it is still connected).
Exacmple of F such that F is bijective and continuous but F^{-1} is not:
Consider parmaterization of circle by $F = (\cos t, \sin t)$ for $t \in [0, 2\pi)$
For a compact set, F is 1-1 and contiuous $\implies$ F^{-1} is continuous $\implies$ F is homeomorphism.

Locally Euclidiean of Dimension n

Def: $(\chi, \mathcal{T})$ is called locally euclean of dimension n if : $\forall x \in X$, there is an open neughbourhood $U \in \mathcal{T}$ and a homeomorphism $h : U \to U'$ with $U' \subset R^n$ open.
As an example, a double cone is not lcoally euclidean at the vertex because the open ball around the vertex is always 3D, so we can't really express it by 2D I think.
More formally, assume it is a manifold and consider the parametrization at the vertex. Removing the vertex, our space is disconnection, while the parametrizing space remains connected after removal of a distinct point (since it is open).

Topological Manifolds

Def: A Hausdorff (points are separable), second countable topological space (space has countable basis) M is said to be an n-dimensional topological manifold if for any point $p \in M, \exists$ a homeomorphism $F : U \to \mathcal{O}, U \subset \mathbb{R}^n$. This hemomorphism is also the local parametrization.

Smooth Manifolds

For abstract manifolds, the manifold may not be in R^n so it may not make sense to talk about differentiability of F. However, we can define defirentiation for transition maps :
Def: An n-dimensional topological manifold M is said to be smooth if there is a collection $\mathcal{A}$ of local parametrizations $F_\alpha : U_\alpha \to \mathcal{O}_\alpha$ such that
(1) $\Cup_{\alpha \in \mathcal{A}} \mathcal{O}_\alpha = M$ so these local parametrizations cover all of M
(2) all transition maps $F_\alpha^{-1} \circ F_\beta$ are smooth on their domains.
Examples: Easy to see that if a topological manifold can be covered by one gloval parametrization (exacmple in $\mathbb{R}^n$), then it is a smooth manifold. The graph of T_f of any continuous f is also a smooth manifold covered by one parametrizations.
Any regular curve $\gamma(t)$ is a smooth manifold of dimension 1.
Any regular surface in $\mathbb{R}^3$ are smooth manifolds.

Tangent spaces

Another way to define these spaces is the collection of vectors which "start" from p, and follow their derivative at p:

$$T_pM = \{\gamma'(0) : (-\epsilon, \epsilon) \to M \text{differentiable with } \gamma(0) = p\}$$

where γ is the parametrised curve $\gamma : \mathbb{R} \to M$.
Remark: This should only be true if manifold is not too abstract, i.e. $M \in \mathbb{R}^n$ since our tangent space lives in $\mathbb{R}^n$. This definition can be extended to abstract manifolds but im too lazy to think about it in the the video

For Abstract Manifolds:

The partial derivative $\frac{\partial}{\partial u_j}(p)$ can be thought as an operator:

$$\frac{\partial}{\partial u_j}(p) : C^1(M, \mathbb{R}) \to \mathbb{R}$$

$$f \to \frac{\partial f}{\partial u_j}(p)(p)$$

where $C^1(M, \mathbb{R})$ denotes the set of all C^1 functions from M to $\mathbb{R}$. Hence we define the tangent spaces

$$T_pM = span\{\frac{\partial}{\partial u_1}(p), \ldots, \frac{\partial}{\partial u_n}(p)\}$$

where the partials are with respect to the local parametrization $F(u_1, \ldots, u_n)$. Surprisingly, the definition does not depend on the choice of parametrization since $\{\frac{\partial}{\partial u_j}(p)\}_{i=1}^n$ happen to be linearly independent, so $dimT_pM = n = dimM$.

Tangent maps

Partial Derivates

Definition 2.29 (Partial Derivatives of Maps between Manifolds). Let $\Phi : M^m \to N^n$ be a smooth map between two smooth manifolds M and N. Let $F(u_1, \ldots, u_m) : \mathcal{U}_M \to \mathcal{O}_M \subset M$ be a smooth local parametrization around p and $G(v_1, \ldots, v_n) : \mathcal{U}_N \to \mathcal{O}_N \subset N$ be a smooth local parametrization around $\Phi(p)$. Then, the partial derivative of Φ with respect to u_i at p is defined to be:

$$\frac{\partial \Phi}{\partial u_i}(p) := \sum_{j=1}^{n} \left.\frac{\partial v_j}{\partial u_i}\right|_{F^{-1}(p)} \frac{\partial}{\partial v_j}(\Phi(p)).$$

Here $(v_1, \dots, v_n)$ are regarded as functions of $(u_1, \dots, u_m)$ in a sense that:

$$(v_1, \dots, v_n) = G^{-1} \circ \Phi \circ F(u_1, \dots, u_m)$$

Remark: The partial derivation $\frac{\partial \phi}{\partial u_i}$ does depend on F, the parametrization of domain space, but not on G, the parametrization of target space.

Chapter 7: Euclidean HyperSurfaces

Regular HyperSurfaces

Def: Consider the local parametrizations that cover the whole space, we require these 3 conditions to holds:

$$(1) F_\alpha(u^1_\alpha, \dots u^n_\alpha) \text{ is smooth}$$

$$(2) F_\alpha \text{ is a homeomorphism}$$

$$(3) \text{ The vectors } \{\frac{\partial F_\alpha}{\partial u^1_\alpha}, \dots, \frac{\partial F_\alpha}{\partial u^n_\alpha}\} \text{ are linearly indepedent}$$

Examples: Graphs are HyperSurfaces, we can easily check the 3 conditions.

z The tow Stereographic parametrizaitons of n-dimensional sphere are hypersurface.

Many results from regular surfaces follow to hypersurface.

Fundamental Forms and Curvatures

First Fundmental Form

Def: For a hypersurface $\sum^n \subset \mathbb{R}^{n+1}$, a 2-tensor g on $\sum$ such that

$$g = \left\langle \frac{\partial F}{\partial u_i}, \frac{\partial F}{\partial u_j} \right\rangle du^i \otimes du^j$$

Remember from Chapter 3, for two linear functionals $A \in V^*, B \in W^*$,

$$A \otimes B : V \times W \to R, (A \otimes B)(X, Y) = A(X)B(Y)$$

for du^i, du^j, is ith component and jth component multiplied together.

$du^i \otimes du^j = \delta_{ij}$ (1 iff $i = j$).

The Matirx $[g]$ is defined similarly, without the tensor products.

This matrix is dependent on the local coordinates, which g is not.

It is also easy to see that this matrix is symmetric, which is always nice to have.

Second Forms

Def: Given a regualr hypersurface in $\mathbb{R}^{n+1}$ with Gauss map ν. We define the second fundamental form of Σ at p to be the 2-tensor:

$$h(X,Y) := -\langle X, D_Y \nu \rangle .$$

Under a local coordinate system $(u_1, \cdots, u_n)$, we denote its component as

$$h_{ij} := h\left(\frac{\partial F}{\partial u_i}, \frac{\partial F}{\partial u_j}\right) = -\left\langle \frac{\partial F}{\partial u_i}, \frac{\partial \nu}{\partial u_j} \right\rangle = \left\langle \frac{\partial^2 F}{\partial u_i \partial u_j}, \nu \right\rangle .$$

Example: For a graph, we can note taht

$$h_{ij} = \langle \frac{\partial^2 F}{\partial u_i \partial u_j}, v \rangle = \frac{\frac{\partial^2 f}{\partial u_i \partial u_j}}{\sqrt{1 + |\nabla f|^2}}$$

And the matrix $[h]$ would then be

$$[h] = \frac{Hess(f)}{\sqrt{1 + |\nabla f|^2}}$$

where $Hess(f)$ is the hessian $= \nabla\nabla f$.
The shape operator S_p tells us how the normal vector v changes in all directions.

Besides the definition of curvature as $\frac{\partial T}{\partial s} = \frac{\partial \theta}{\partial s}$, intuitively we can also say the curvature is the $\frac{1}{\text{radius}}$ of the circle that just touches that point.
Some more curvature formulations :

$$\frac{\partial L}{\partial t} = -\int k^2 ds = -\int k d\theta$$

Hamil and Cage paper - section 4

A strictly conves closed curve has some nice properties that we could use. Firstly, strictly convex means that we can parametrize it by θ - the angel from a fixed straight line (usually the x axis). This is because in a strictly convex curve, this angle is strictly increasing (or decreasing) as you go around the close loop. This formally means that the curvature is never 0.
Closed loop means that we travel a full 2π degrees, and that we end up in the same point. In particular $\int_0^{2\pi} T(\theta) d\theta = 0$ where $T(\theta)$ is the tangent.

4.1 and 4.2

The curve shortening process is a flow where each shape follows the direction of it's curvature. So if at a given time $t = t_0$ we have a curvature points inwards, that point to go inwards at the rate of the curvature. Since we are dealing with convex curves, we will only have inwards curvature.
Section 4.1 of the paper proved that the curve shortening process for the convex curves is equivalent to an IVP for a PDE. This means that our curve shortening flow only depends on the initial constructior an no other facors. The paper also proved previous that during a process, a convex curve remains convex by showing that the smallest curvature is always increasing (so it never goes "inside" as it is initially positive).

4.3

Def: Median curvature (different from mean curvation) k^* is defines as

$$k^* = sup\{b|k(\theta) > b \text{ on an interval of lenght atleast } \pi\}$$

This is useful for coming up with certain estimates:
Geometric Estimate bounds median curvature by ratio of arc lenght to area.
Integral Estimate states that $k^*(t)$ bounded on $[0, T) \implies \int_0^{2\pi} \log k(\theta, t) d\theta$ is also bounded on $[0, T)$.
Pointwise Estimate gives a strong bound for the opposite direction. $\int_0^{2\pi} \log k(\theta, t) d\theta$ is bounded on $[0, T) \implies k(\theta, t)$ is uniformly bounded on $S^1 \times [0, T)$.
Remarks: Boundedness of $\int_0^{2\pi} \log(k(theta, t)) d\theta \not\Longrightarrow$ boundedness of Boundedness of $\int_0^{2\pi} k(theta, t) d\theta$. As an example, let $k(\theta, t) = e^{\frac{1}{x}}$.
4.3.1 : Given we have a proper curvature (satisfied 4.14), if we bound t such that the area enclosed is bounded away from 0 (so it does not collapse), then curvature $k(\theta, t)$ is uniformly bounded.
Since it does not collapse, we have a smallest ciricle who's inverse radius gives the curvature, hence was have na upper bound (basically geometric estimate).
To prove uniformity, by integral estimate since $k^*(t)$ is bounded, the given log iintegral is bounded.
Since log integral is bouded, by pointwise estimate, we have that k is uniformly bounded.
Cool lemma: if the log integral is bounded, then for any $\delta > 0, \exists C$ such taht $k(\theta, t) \leq C$ except on intervals of lenght $\leq \delta$.

Section 4.4

4.4.1 shows that k bounded $\implies \frac{\partial k}{\partial \theta}$ is also bounded as our PDE shows that $\frac{\partial k}{\partial \theta}$ grows at most exponentially.
Additionally, if k is bounded $\implies \frac{\partial k}{\partial \theta}$ is bounded alos $\implies \int_0^{2\pi}(k'')^4$ is bounded. The proof is just by using the evolution equation.

This section assentially proves that if the areas is bounded away from 0, $k^P(i)$ are all bounded for $i \in \mathbb{N} \cup 0$.
This analysis can also give a bound with respect to time (given area does not go to zero). Suppse the solution exists of $[0, T)$ with area bounded away from 0, then k has a limit $t \to T$ which is C^∞ and we can extend the solution after T.