Geometric Flows

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Abstract

In this report we study the Curve Shortening Flow (CSF). We first see why convex shapes are special, then we study some properties of the flow - namely it's affect on the arclenght, area and the overall shape, and finally we conclude with some time bounds on CSF to show that inscribed curves remain inscribed.

Properties of Convex Curves

We first turn our attention to strictly convex curves, and see why they may be intersecting to study in the first place.

Parametrization of strictly Convex Curves

One interesting property is that we can parametrize strictly convex curves by just one parameter θ which represents the angle the tangent vector makes with a fixed straight line (usually the x-axis).
The intuition is that if you travel along the the curve, the vector T is always moving along the same (clockwise or anticlockwise) direction since the normal vector N is always pointing inwards.
For a closed convex curve, we can easily see that θ is in the domain $[0, 2\pi)$. In particular we have

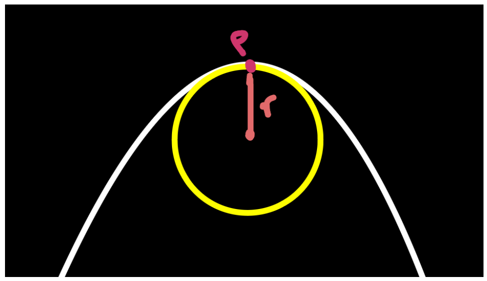
$$\int_0^{2\pi} T(\theta)d\theta = 0$$

Curvature on Convex Curves

The curvature $k(s)$ is defined as

$$k(s) = \left|\frac{dT(s)}{ds}\right| = \left|\frac{d\theta}{ds}\right|$$

where $T(s)$ is the arclength parametrization.
One geometrically intuitive way to also understand it is with the follow figure:



The curvature at point p is the inverse radius of the circle which touches p, i.e. $k = \frac{1}{r}$.
Guass - Bonnet Theorem for plane curves shows us that for any closed curve, we have total curvation as

$$\int_0^L k(s)ds = 2\pi n$$

Here n is actually the winding number - or how many times our curve twists and turns.
We show that for convex closed curve we have $n = 1$.
Lemma 1: For any convex curve (not necessarily closed) , our total curvature is at most 2π.

$$\begin{aligned}\int_0^L k(s)ds &= \int_0^L \left|\frac{d\theta}{ds}\right| ds \\ &= \theta(L) - \theta(0) \\ &\geq 2\pi\end{aligned}$$

The last line follows from the fact that θ is monotonous with domain $[0, 2\pi)$ (as we discussed in previous section), hence the maximum possible change in θ is 2π. ■

By Gauss Bonnet, we hvae that for closed surves, total curvature is a multiple of 2π. Combining with Lemma 1, we get that for closed convex curves, the total curvature is exaclty $\boxed{2\pi}$.

Curve Shortening process

The curve shortening process describes the flow when at each point on the curve, the curve moves towards the vector N at the rate of $k(p)$, the curvature at that point. Formally, this flow can be describe by the equation

$$\frac{\partial \gamma}{\partial t} = -kN$$

Hence we can see that to study this flow, we should consider smooth graphs with continuous curvature.
This website : `https://a.carapetis.com/csf/` provides a simulation of the flow for any given closed curve.
As an example, a circle of radius r, has constant curvature $\frac{1}{r}$ with N pointing towards the center. This means that the circle is shrinking towards the center at an increasing rate (w.r.t time). This shrinking is infact a global property of this flow!

Analysing the Length

We want to show that the length is always decreasing, hence any closed curve will be shrinking.

$$\begin{aligned}
\frac{d}{dt}L &= \int_I \frac{d}{dt}\left\|\frac{dX(v,t)}{dv}\right\| dv \\
&= \int_I \frac{\left\langle \frac{d^2X}{dvdt}, \frac{dX}{dv}\right\rangle}{\left\|\frac{dX}{dv}\right\|} dv \\
&= \int_I \left\langle \frac{d^2X}{dvdt}, T_*\right\rangle dv \\
&= \int_I \left\langle \frac{d(kN)}{dv}, T_*\right\rangle dv \\
&= \int_I \left\langle \frac{dk}{dv}N, T_*\right\rangle + k\left\langle \frac{dN}{dv}, T_*\right\rangle dv \\
&= \int_I k\left\langle \frac{dN}{dv}, T_*\right\rangle dv
\end{aligned}$$

Using Frenet equations, note that

$$\left\langle \frac{dN}{dv}, T\right\rangle = \left\langle -kT\frac{ds}{dv}, T\right\rangle = -k\frac{ds}{dv}$$

Hence we arrive at the rate of change of length as

$$\frac{dL}{dt} = -\int_{\gamma(t)} k^2 ds$$

which is clearly decreasing. ■
Remark: Reference Hamilton and Cage 3.1.1.

Analysing the Area

Given the above results, we would also expect the area to decrease over time.

$$A(t) = \frac{1}{2}\int_{\gamma(t)} (xdy - ydx)$$

Taking the derivative

$$\frac{dA}{dt} = \frac{1}{2}\int_{\gamma(t)} (\frac{\partial}{\partial t}(xdy - ydx))$$

By our curve flow, $\frac{\partial \gamma}{\partial t} = -kN$. Notice that the term $xdy - ydz$ corresponds to the line integral on the curve. So, using divergence theorem, we use the fact taht the velocity of the curve along wht normal direction is exactly the curvate with the divergence of the velocity field $-kN$.

$$\frac{dA}{dt} = -\int_{\gamma(t)} kds$$

which is clearly decreasing.
Infact, by our results shown above, on a closed convex curve, we have that $\frac{dA}{dt}$ is exactly $\boxed{-2\pi}$.

Remark: Reference Hamilon and Cage 3.1.7.

Invariance of convex curves

Since our flow agrees with the curvature (moves along the normal which is always pointing inwards), we would expect a convex curve to remain convex during the flow until it shrinks down to a point. This is indeed the case!
By Cage and Hamilton 3.1.6, we have the equation

$$\frac{\partial k}{\partial t} = \frac{\partial^2 k}{\partial s^2} + k^3$$

Since the space domain is compact, k must achieve a minimum. Hence, consider the point p with minimum curvature so that we have $\frac{\partial k^2}{\partial s^2} > 0$. Hence we get that

$$\frac{\partial k}{\partial t} > k^3 > 0$$

But since our initial curve was convex, i.e. $k > 0$, and we have shown that signed curvature is always increasing, we always have positive signed curvature which is sufficent to conclude that it remains convex.
Additionally, if $k = 0$, i.e. we have a straight line segment, we can see that

$$\frac{\partial k}{\partial t} > k^3 \geq 0$$

At the line segment, $k = 0$, but $\frac{\partial k}{\partial t} > 0$, hence a convex curve will quickly (at the next instant) become a strictly convex curve.

Flow of non-convex curves

We have studied some properties about how the flow affects convex curves. For non-convex curve, we still have the result that the lenghts and the areas are always shrinking. Hence, just like convex curves, non-convex curves also shrink down to a point. Therefore, we would expect, that a non-convex curve "smooths out eventually" into a convex curve.
This is exactly the statement of Cage-Hamilton-Grayson Theorem.
Cage-Hamilton Theorem: Consider the IVP for curve shortening flow where the initial curve γ_0 is convex, embedded and closed. Then, the curve has a unique solution which is uniformaly convex for each t as it shrinks to a point.
Grayson's Theorem Convergence to a round point is given by the existence of a unique point $x_0 \in \mathbb{R}^2$ such that the rescaled flows

$$\gamma_t^\lambda = \lambda \cdot (\gamma_{T+\lambda^{-2}t} - x_0)$$

converge of $\lambda \to \infty$ to the round circle $\{\partial B_{\sqrt{-2t}}\}_{t \in (-\infty,0)}$ which shrinks.
If the initial curve γ_0 is closed embedded cruve, then the curve shortening flow converges to a round point.

Hence Grayson generalized Cage-Hamilton by proving that non-convex curves eventually become convex so that we can apply the same analysis to it eventually.

Bounds on Time

We have shown that any closed curve shrinks to a point p. In fact, we have shown that the time T taken for the curve to reach singularity is finite since we showed that

$$\frac{dA}{dt} = -2\pi$$

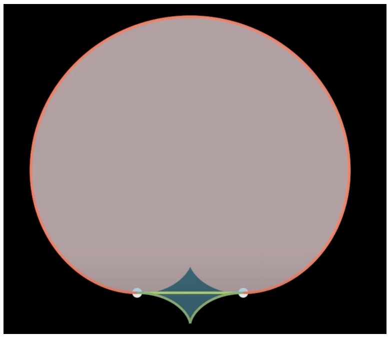
Hence, we can easily see $T \leq \frac{A_0}{2\pi}$.
In this section, we try to create a a bound on T with respect to the arclength l instead.

Bounds on Arclength

Since we know the bound of T with respect to the initial area A_0, we will try to create a bound on l_0, the initial arclength, with respect to area.
In order words, given a fixed arclength, we want to find the shape with the greatest possible area.

Our shape much be convex

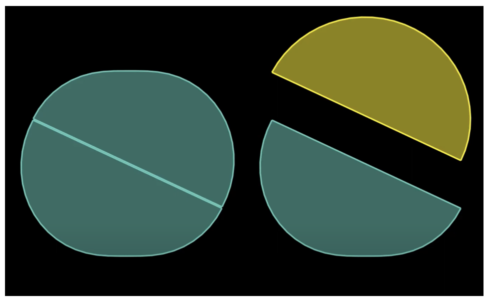
Firstly, we can not have any concave shape.



Assume there exists a concave shape with the greatest area. Then by defintion there exists 2 points a, b such that the line segment joining a, b is not entirely within the shape.
We simply reflect the segment of the curve by this line segment, so for the same arclength, we get a greater area which is a contradition. ■

Focusing on one half

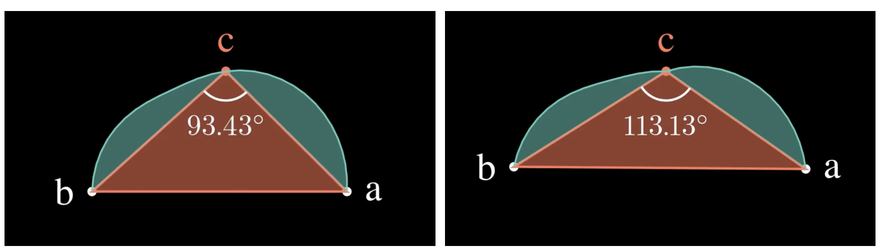
We now focus on only convex shapes. Infact, within convex shapes we just focus on half of the shape due to the symmetry.
We travel around our curve for half the arclenght and split up our curves. Assume that one of the halves has greater area, then we can just duplicate and join it together to get a greater area.



If the blue half has more area, we just join to get a "better" figure

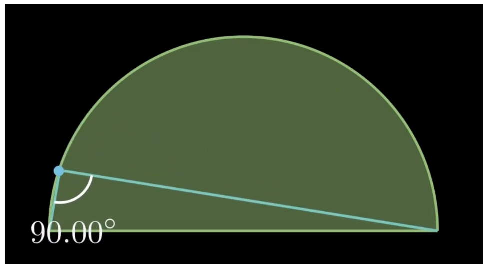
Focusing on half curves

Now we take any "half curve" with endpoints a, b and take any arbitrary point c on the curve. Now consier the orange triangle as shown in the figure, and notice that we can vary the angle $\angle ACB$ while keeping the arclenght constant.



We can varry the angle while keeping the length same

So now, our problem just becomes finding the angle which maximizes the area of this traingle. It is easy to see that $\angle ACB$ must be $90°$.
Therefore, our shape must be such that for any given $c, \angle ACB = 90° \implies$ our half curve is a semi-cricle $\implies$ our ideal shape is a circle. ■



The isoperimetric property

So, for a fixed arclength l, a circle gives us the greatest area A, with the ratio
$$\frac{A}{l} = \frac{\pi r^2}{2\pi r} = \frac{r}{2}$$
Here r is the radius of the circle we get with arlength l. We wish to get rid of this r and so we will focus on l^2.
Notice that for a given l^2, the circle would still give us the greatest area.

$$\frac{A}{l^2} = \frac{\pi r^2}{4\pi^2 r^2} = \frac{1}{4\pi}$$

Finally, we get the isoperimetric bound for the arclength

$$l^2 \geq 4\pi A$$

where the equality holds only for circle.

Bounds on time

Now that we have related A_0 to l_0, we arrive at the following bound :

$$T \leq \frac{A_0}{2\pi} \leq \frac{l_0^2}{8\pi^2}$$

As a remark, we can notice that for a fixed length, the circle takes the longest time to reach a point given by

$$T = \frac{l_0^2}{8\pi^2} = \frac{4\pi^2 r^2}{8\pi^2} = \frac{r^2}{2}$$

We can also relate this to curvature k, since a circle has curvature $k = \frac{1}{r}$

$$T = \frac{r^2}{2} = \frac{1}{2k^2}$$

Bounding with curvature

This, in fact gives us an upper bound for strictly convex corve

$$T \leq \frac{1}{2k_{min}^2}$$

where k_{min} is the minimum curvature at time $t = 0$.
Intuitively, this means that if we find a circle of radius r such that our graph γ is inscribed within the circle, then the graph will reach singularity before the circle.
Doing the same analysis, we a lower bound on the time by taking k_{max} at $t = 0$

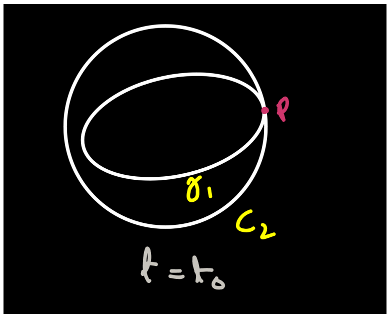
$$T \geq \frac{1}{2k_{max}^2}$$

Intuitively, this also means that if we find a circle of radius r' such that our circle is inscribed within the graph γ, then the circle will reach singularity before the graph.
So far, for the graph γ, we have found 2 circles C_1, C_2, such that

$$T(C_1) \leq \gamma \leq T(C_2)$$

Where $T(\gamma)$ is the time taken for γ to reach singularity.

We can actually make the bound tigther by claiming that any C_1 that is inscribed within γ will finish first, and any C_2 which inscribes γ will finish later.
We present the proof of the later case, the proof for former case follows similarly.
Assume we have a C_2 which incribes the graph γ and we consider the minimum $t = t_0$ such that $\exists$ a point p on both γ and C_2.

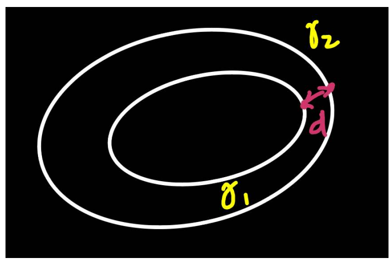


It is easy to see that the curvature $k_\gamma(p) \geq k_{C_2}(p)$ hence this point can not shrink slower than the circle $\implies$ it will never "cross" the circle $\implies$ it remains inscribed inside the circle. ■

Inscribed Graphs

We now extend this idea to incribed graphs.
Assume we have two graphs γ_1, γ_2 such that γ_1 isinscribed within γ_2. Formally, "inscribed" here means that

$$d(t) = \inf ||\gamma_2(t) - \gamma_1(t)|| \geq 0$$



Assume at some point $t_0, d(t_0) = 0 \implies \exists$ points $p_1 \in \gamma_1, p_2 \in \gamma_2$ such that $p_1 = p_2$.
We use a similar idea as previous section, but now we use the maximum principle to argue that at this point, $k_{\gamma_1}(p_1) \geq k_{\gamma_2}(p_2)$. Consider the equations of evolution, these two points can not "cross" each other $\implies$ γ_1 remains inscribed inside γ_2.

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This study has also made me interested in the generalization of CSF to

higher dimensions called the Mean Flow Curvature and I hope to study that in some future. There is also some analysis on the forming of multiple singularities which is an interesting area of study.

Citations

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